S. Keel - 29/1/13 - Mirr. symmetry for CYs w/ max boundary I

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<u>Ex</u>: $U = (\mathbb{C}^\times)^N$, $V = \Gamma(U, \mathcal{O}_U) = \bigoplus_{q \in \mathbb{Z}^n} \mathbb{C} z^q$ has a "canonical" basis

eg. $f$ nowhere zero $\iff f = \lambda z^q$, $\lambda \in \mathbb{C}^\times$, $q \in \mathbb{Z}^n$.

<u>Conj</u> (GHK): Same for any affine CY $U$ with maximal $\partial$, ie.
- ring of functions $V$ has a canonical "basis"
- structure constants for multn in this basis are defined by counts of rational curves on the mirror.

<u>Def</u>: $U$ smooth is called CY if $\exists\, \omega \in \Gamma(U, \Lambda^N T_U^*)$ with at most simple poles on any compactification divisor $D$ of $U \underset{\text{open}}{\subset} Y$ : $\mathrm{val}_D(\omega) \geq -1$ and moreover $\omega$ with this property is unique up to scaling.

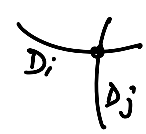
eg. $(Y, D)$ simple normal crossings compactifn of $U$ and $K_Y + D \equiv 0$.
$\Rightarrow U = Y \setminus D$ is CY.

Call $(Y, D)$ a "minimal model" of $U$

(typically $\exists\, \infty$ many equally good such $(Y,D)$'s for given $U$).

"<u>maximal boundary</u>" := want $D$ to contain a "zero stratum".

ie. $\exists\, p \in D \subset Y$ looks like
$0 \in \{z_1 \dots z_N = 0\} \subset \mathbb{C}^N_{z_1, \dots, z_N}$




<u>Key def$^n$</u>: $U^{trop}(\mathbb{Z}) = \{ v : Q(U) \setminus \{0\} \to \mathbb{Z} \,/\, v(\omega) < 0 \} \cup \{0\}$ valuations on rat! functions "supp$^d$ at $D$ for <u>some</u> $D$" (since $v(\omega) < 0$)

rational f$^{ns}$

$= \{ (E, m) \,/\, E$ is a boundary divisor in <u>some</u> minimal model, $m > 0 \} \cup \{0\}$

<u>Ex</u>. <u>cluster varieties</u> : eg. Fomin-Zelevinsky:

$G$ semisimple, $G/U \to G/B$ homog bundle,
"flag manifold

$$X(G/B) = \bigoplus_{L \in \mathrm{Pic}(G/B)} H^0(G/B, L)$$ cluster algebra

In fact canonical basis is known: Lusztig basis

$$\Gamma(G/U, \mathcal{O}) = \bigoplus_{L \in \mathrm{Pic}(G/B)} H^0(G/B, L)$$

↑ $\theta$-function conj: says this has a canonical basis.

get basis for every rep. .....

Goal: <u>do this without representation theory</u>

Let $V$ = vector space w/ basis $U^{trop}(\mathbb{Z})$.

....

**Thm:** This works for $U$ of dim 2, and for lots of cluster varieties eg. any cluster variety w/ acyclic seed.

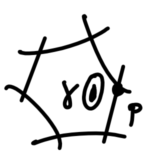
- $0 \in U^{trop}(\mathbb{Z})$ is canonical: you can scale valuations.

$\vartheta_0$ special, $V \xrightarrow{\text{trace}} k$ valuation

$f \longmapsto$ coeff of $f$ at $\vartheta_0$.

**Claim:** $\exists$ canonical $m$-point functions $\underbrace{V \times \dots \times V}_{m} \to k$

**Claim:** $\exists$ canonical $\gamma \in H_N(U, \mathbb{Z})$



← max $\partial$. where $p \in D \subset Y$ looks like $0 \in \{z_1 \cdots z_N = 0\} \subset \mathbb{C}^N$

look at $\gamma = [(S^1)^N_\varepsilon]$
$= \{|z_1| = \dots = |z_N| = \varepsilon\}$

**Conj:** $[\gamma] \in H_N(U, \mathbb{Z})$
1) is indep't of chosen 0d stratum $p$
2) is indep't of chosen minimal model $Y$.

**Thm:**
1) true in all dim. (= Shokurov connectedness)
2) true in dim. 2 & for cluster vars.

Pick $\omega \in \Omega^{n,0}(U)$ s.t. $\int_\gamma \omega = 1$.

....

<u>Trace</u>: $V = \Gamma(U, \mathcal{O}_U) \to \mathbb{C}$

$$f \longmapsto \int_\gamma f\omega \qquad \text{(NB } f\omega \text{ is closed)}$$

and $V^N \to \mathbb{C}$

$$\langle f_1, \dots, f_N \rangle := \mathrm{Tr}(f_1 \cdots f_N) = \int_\gamma (f_1 \cdots f_N)\,\omega .$$

<u>Conj.</u> (<u>Thm in dim 2</u>):

The 2-point pairing $\Gamma(U, \mathcal{O}_U) \times \Gamma(U, \mathcal{O}_U) \to \mathbb{C}$ is nondegenerate.

ie. $\langle f, \cdot \rangle : V \to \mathbb{C}$ determines $f$.

<u>Note</u>: product structure on $\Gamma(U, \mathcal{O}_U)$ is determined by Trace, $\langle \cdot, \cdot \rangle$, and $\langle \cdot, \cdot, \cdot \rangle$.

Indeed: $\langle fg, \cdot \rangle = \langle f, g, \cdot \rangle$

so nondegeneracy $\Rightarrow$ $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot, \cdot \rangle$ determine mult$^n$.

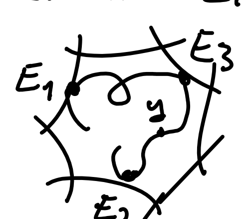
<u>Now</u>: to build mirror of $U$, consider $V = \bigoplus_{q \in U^{trop}(\mathbb{Z})} k \cdot \theta_q$ and equip it with the trace $V \to k$ given by coeff$^t$ of $\theta_0$.

If we think Spec $V$ = mirror of $U$ and there's a mult$^n$ then $V$ should have a product structure, hence $k$-point functions (which should come from enumerative geometry)

$\to$ need to define $\langle \theta_{q_1}, \dots, \theta_{q_N} \rangle \in k$ for all $q_i = (E_i, m_i)$ compactif$^n$ divisors

Namely, given $(Y, D)$ "large enough" compactif$^n$ so all $E_i \subset D$,

$\langle \theta_{q_1} \dots \theta_{q_N} \rangle$ = count of rational curves in $Y$ which only touch $E_i$ at one point, w/ order $m_i$, and through a general pt $y \in Y$



Je count maps $(\mathbb{P}^1, p_1 \dots p_n, x) \xrightarrow{\varphi} Y$

$\varphi(x) = y$ given general pt

$\varphi^{-1}(D) = \{p_1 \dots p_n\}$ as a set

$\cap\ E_i$ w/ multiplicity $m_i$ at $\varphi(p_i)$.

dim. count of such maps (using $D = -K_Y$) is $n+1-3 = \dim \mathcal{M}_{0,n+1}$

$\rightsquigarrow$ if we fix also the modulus of $(x, p_1 \dots p_n)$, eg. for $n=3$ the cross-ratio, then expected dim is zero and we can count!

$\triangle$ in fact one really has to fix class $\alpha \in H_2(Y, \mathbb{Z})$ rep$^d$ by the curve.

$\alpha = [C] \in NE(Y) \subset H_2(Y,\mathbb{Z})$ Mori cone

$\rightsquigarrow$ given $(E_1, m_1), \dots, (E_n, m_n)$ and $\alpha$, get a count of curves.

$$\Rightarrow \quad V \times \dots \times V \longrightarrow R = k[NE(Y)]$$

$$\Big(V = \bigoplus_{q \in U^{trop}} R.\,\theta_q\Big) \qquad = \bigoplus_{C \in NE(Y)} k.\, z^{[C]}$$

Conj: Take $(Y, D)$, $K_Y + D = 0$, assume has 0 stratum and $D$ supports an ample divisor.

$(\Rightarrow U = Y - D$ affine CY w/ max $\partial$ and $NE(Y)$ finite polyhedral cone)

Then (1) $\exists!$ $R = k[NE(Y)]$-algebra structure on $V = \bigoplus_{q \in U^{trop}(\mathbb{Z})} R\,q$

s.t. • $1_V = \theta_0$

• $\mathrm{Coeff}_{\theta_0}(f_1 \dots f_n) = \langle f_1, \dots, f_N \rangle$

(2) $$\begin{array}{ccc} \mathrm{Spec}(V) & \xrightarrow[\text{flat}]{} & \mathrm{Spec}\, R \\ \cup & & \cup \\ \mathfrak{X} & \xrightarrow{\pi} & \mathrm{Toric}_{\mathrm{Pic}(Y)} \end{array}$$ $= $ toric var. assoc to $NEF(Y)$

$\uparrow$ family of affine CYs (mirror family of $U$)

fiber $= U'$

(3) $V = SH^0(U)$ with sympl. cohomology product

(4) Take $U'$ = fiber of mirror family, $U'$ affine CY w/ max $\partial$
so run program again $\Rightarrow$ get

$$\mathrm{Spec}(V') \longrightarrow \mathrm{Toric\ va}(NEF(Y))$$

↑ this is the univ. deformation space of $U$

(5) If $U$ is holomorphic symplectic then mirror is canonically isom. to $U$

(eg dim 2, unimodular cluster)

<u>Thm:</u> This works for $\dim U = 2$, and for lots of clusters, eg. acyclic seeds.

↑ (or: when $\exists$ maximal green mutation sequence in sense of Keller)

<u>NB:</u> didn't explain what coeff of $\theta_r$ in $\theta_p \cdot \theta_q$ is, but instead we gave $\langle \theta_p \theta_q, \theta_s \rangle$ $\forall s$.